Derivative of a Bernstein Polynomial

With $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$, i = 0…n , we get

$$\begin{aligned}
\frac{d}{dt} B_i^n(t) &= \binom{n}{i} i t^{i-1} (1-t)^{n-i} + \binom{n}{i} t^i (-1)(n-i)(1-t)^{n-i-1} \\
&= \frac{n!}{i!\,(n-i)!} i t^{i-1} (1-t)^{n-i} - \frac{n!}{i!\,(n-i)!} (n-i) t^i (1-t)^{n-i-1} \\
&= n \left[\frac{(n-1)!}{(i-1)!\,(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i!\,(n-i-1)!} t^i (1-t)^{n-i-1} \right] \\
&= n \left[\binom{n-1}{i-1} t^{i-1} (1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{n-1-i} \right] \\
&= n (B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) \qquad \text{with} \qquad B_{-1}^{n-1} = B_n^{n-1} = 0
\end{aligned}$$

Now we show that a Bernstein Polynomial has just one maximum in [0,1], at t=i/n:

$$\begin{aligned}
\frac{d}{dt} B_i^n(t) &= \binom{n}{i} i t^{i-1} (1-t)^{n-i} + \binom{n}{i} t^i (-1)(n-i)(1-t)^{n-i-1} \\
&= \binom{n}{i} [i t^{i-1} (1-t)^{n-i} - t^i (n-i)(1-t)^{n-i-1}] \\
&= \binom{n}{i} t^{i-1} [i (1-t)^{n-i} - t(n-i)(1-t)^{n-i-1}] \\
&= \binom{n}{i} t^{i-1} (1-t)^{n-i-1} [i(1-t) - t(n-i)]
\end{aligned}$$

$\binom{n}{i} t^{i-1} (1-t)^{n-i-1}$ can be zero in t=0 or t=1, additionally we are left with:

$$\begin{aligned}
& i(1-t) - t(n-i) = 0 \\
\Rightarrow\ & i - it = tn - it \\
\Rightarrow\ & i = nt \\
\Rightarrow\ & t = \frac{i}{n}
\end{aligned}$$

So we can have a maximum at three positions, so let's check them:

Case 1: t = 0 $\quad B_i^n(0) = \binom{n}{i} 0^i 1^{n-i}$. If i ≠ 0 the term is 0; if i = 0 we have $\binom{n}{0} 0^0 1^n = 1$ which is a maximum cause $0 \leq B_i^n(t) \leq 1$ for t ∈ (0,1). But that fits with $t = \frac{i}{n} \quad t = \frac{0}{n} = 0$.

Case 2: t = 1 $\quad B_i^n(1) = \binom{n}{i} 1^i 0^{n-i}$. If i ≠ n the term is 0; if i = n we have $\binom{n}{n} 1^n 1^0 = 1$ which is a maximum cause $0 \leq B_i^n(t) \leq 1$ for t ∈ (0,1). But that fits with $t = \frac{i}{n} \quad t = \frac{n}{n} = 1$.

Case 3: For all other t, t ∈ (0,1), we have an extremum at $\frac{i}{n}$. To prove that it's a maximum we can examine the second derivative or we argument another way:
The cases t = 0 and t = 1 are already checked and $B_i^n(0) = 0$, $B_i^n(1) = 1$ for i < 0 < n and $B_i^n(\frac{i}{n}) > 0$ for i < 0 < n, so it must be a maximum.

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