Derivation of the two-dimensional dot product

Content

1. Motivation

This article presents the derivation of the two-dimensional dot product. Especially if you have asked yourself following questions, then you're exactly right here:

- Where does the connection to the cosine come from?
  In other words: Where does the definition $\mathbf{a} \cdot \mathbf{b} = |a| * |b| * \cos\alpha$ come from?
- Why is the two-dimensional dot product calculated by $\mathbf{a} \cdot \mathbf{b} = a_x * b_x + a_y * b_y$?

Note that vector are written as bold small letters, e.g. vector a is written by $\mathbf{a}$. The length of vector a is written as $|\boldsymbol{a}|$.

2. Derivation of the dot product in R^2

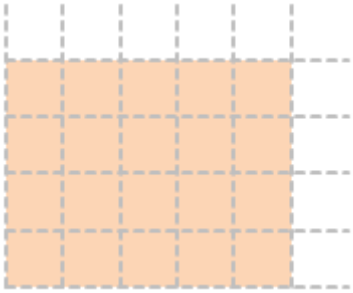
The derivation is done step by step, from trivial obvious facts to the definition of the dot product in following order:

- Definition of the area of a rectangle
- Definition of the area of a right-angled triangle
- Definition of the Pythagorean theorem
- Definition of the area of a general triangle using the law of cosines
- Derivation of the dot product from the law of cosines

2.1. Area of a rectangle

The calculation of the area of a rectangle is so obvious that a formal derivation is discarded. Given the width a and height b, the area if just the number of square units inside the rectangle, thus area_rectangle = a * b.

This can be easily seen from following figure by just counting the number of square units.

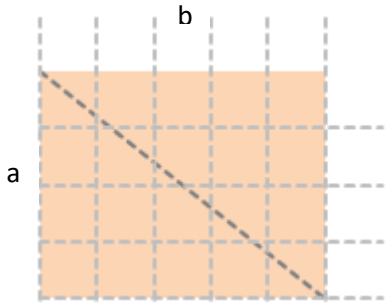


Example of a rectangle with side lengths a = 5, b = 4 and area a*b = 20.

2.2. Area of a right-angled triangle

A right-angled triangle is a triangle where one angle is a right angle. The side opposite to the right angle is called hypotenuse, the other two sides are called legs. Defining the two legs as a and b, the area is calculated as following:
area_triangle = ½ * a * b.

This is directly derived from the area of a rectangle:



Example of a right-angled triangle with side lengths a = 5, b = 4

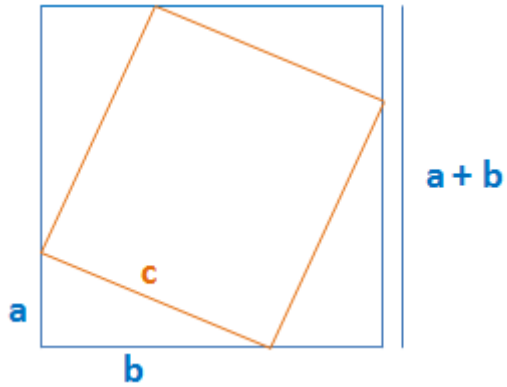
It's easy to see that two equal right-angled triangles with legs a and b build a rectangle with width a and height b. So if two equal triangles make a rectangle with area = a * b, then one triangle has an area of ½ * a * b.

2.3. Pythagorean Theorem

The Pythagorean theorem provides a relation of the three side of a right-angled triangle. It states that the square of the hypotenuse c is equal to the sum of the squares of the legs a and b:

$c^2 = a^2 + b^2$

This can be derived geometrically using following figure:



Small orange square inserted into bigger blue square

The large blue square has side length of a + b, so its area is:
$area1_{bluesquare} = (a + b) * (a + b) = a^2 + 2ab + b^2$

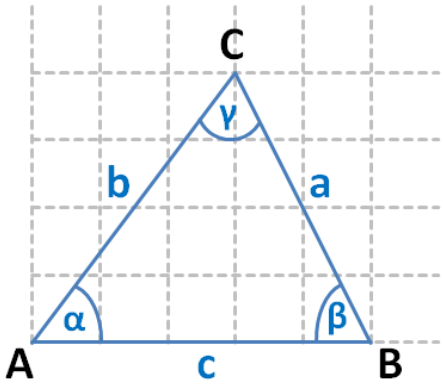
Another way to define the area is the sum of the smaller orange square and the four small blue triangles around:
$area_{orangesquare} = c * c$
$area_{bluetriangle} = {}^1/_2 * a * b$
$area2_{bluesquare} = area_{orangesquare} + 4 * area_{bluetriangle}$

So put $area1_{bluesquare}$ and $area2_{bluesquare}$ on the same level and dissolve it:
$a^2 + 2ab + b^2 = c^2 + 4 * ({}^1/_2 * a * b)$ (solve the brackets)
$a^2 + 2ab + b^2 = c^2 + 2ab$ (cancel out 2ab on both sides)
$a^2 + b^2 = c^2$

2.4. Area of a general triangle using the law of cosines

The law of cosines can be derived using the distance formula between two points that is completely based on the Pythagorean theorem.

The law of cosines relates the three side of any triangle to the cosine of one angle. In other words, if (the length of) two sides of any triangle and the angle between these two sides are known, then (the length of) the third side can be calculated.



E.g. if the sides b and c and the angle α are known, then side a can be calculated.

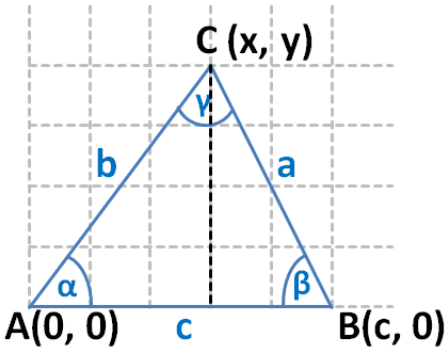
Summarized, the law of cosines for all three cases is:

$$a^2 = b^2 + c^2 - 2 * b * c * \cos\alpha$$

$$b^2 = a^2 + c^2 - 2 * a * c * \cos\beta$$

$$c^2 = a^2 + b^2 - 2 * a * b * \cos\gamma$$

To derive this formula, the Cartesian coordinate system and the distance formula are used:



The point A is placed at the origin. Point B has x-coordinate of length of side c, thus coordinate (c, 0).
Point C has coordinate (x, y) where $x = b * \cos\alpha$ and $y = b * \sin\alpha$.
The side a has length of segment BC that is calculated using the distance formula as
$a = \sqrt{(x-c)^2 + (y-0)^2}$.

Now substitute $x = b * \cos\alpha$ and $y = b * \sin\alpha$ and dissolve the equation:

$$a^2 = (b * \cos\alpha - c)^2 + (b * \sin\alpha - 0)^2$$

$$a^2 = b^2 cos^2\alpha - 2bc\cos\alpha + c^2 + b^2 sin^2\alpha$$

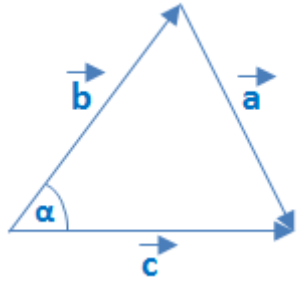
$$a^2 = b^2(cos^2\alpha + sin^2\alpha) + c^2 - 2bc\cos\alpha$$

$$a^2 = b^2 + c^2 - 2 * b * c * \cos\alpha$$

2.5. Derivation of the dot product from the law of cosines

Finally we reach the dot product that is going to be derived from the law of cosines.

Let's have again a look at our triangle, but note that the sides are now treated as vectors:



Thus the length of triangle side a is the length of vector **a** and that is a = $|\boldsymbol{a}| = \sqrt{{a_x}^2 + {a_y}^2}$. The vectors are defined as $\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$ and **c** $= \begin{pmatrix} c_x \\ c_y \end{pmatrix}$. The vector **a** can be written as **c** – **b**. Using the law of cosines, rearrange the equation step by step by replacing **a** and dissolve it:

$$|\boldsymbol{a}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 - 2 * |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

$$|(\boldsymbol{c} - \boldsymbol{b})|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 - 2 * |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

$$(c_x - b_x)^2 + (c_y - b_y)^2 = {b_x}^2 + {b_y}^2 + {c_x}^2 + {c_y}^2 - 2 * |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

$${c_x}^2 - 2b_xc_x + {b_x}^2 + {c_y}^2 - 2b_yc_y + {b_y}^2 = {b_x}^2 + {b_y}^2 + {c_x}^2 + {c_y}^2 - 2 * |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

$$-2b_xc_x - 2b_yc_y = -2 * |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

$$b_x * c_x + b_y * c_y = |\boldsymbol{b}| * |\boldsymbol{c}| * \cos\alpha$$

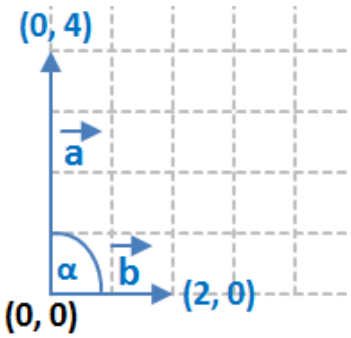
Wow! That our definition of the dot product!

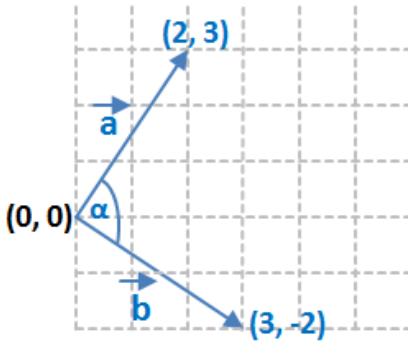
Renaming **b** and **c** to **a** and **b**, the dot product is defined as $\mathbf{a} \cdot \mathbf{b} = |\mathrm{a}| * |\mathrm{b}| * \cos\alpha$, so from above equation we see that $\mathbf{a} \cdot \mathbf{b} = \mathrm{a_x} * \mathrm{b_x} + \mathrm{a_y} * \mathrm{b_y}$. That's exactly what was to be shown.

3. Geometric Interpretation

3.1. Basics

Have a look at the following figures where **a** and **b** are orthogonal vectors, thus $\alpha = 90°$.

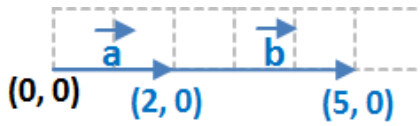




The dot product for the left figure is $\mathbf{a} \cdot \mathbf{b} = a_x * b_x + a_y * b_y = 0 * 2 + 4 * 0 = 0$.
That's the same as $|a| * |b| * \cos\alpha = 4 * 2 * \cos 90° = 0$.
The dot product for the right figure is $\mathbf{a} \cdot \mathbf{b} = a_x * b_x + a_y * b_y = 2 * 3 + 3 * (-2) = 0$.
That's the same as $|a| * |b| * \cos\alpha = \sqrt{2^2 + 3^2} * \sqrt{3^2 + -2^2} * \cos 90° = 0$.

➔ If the dot product is zero, then the vectors are orthogonal.

What happens if the two vectors have the same direction?

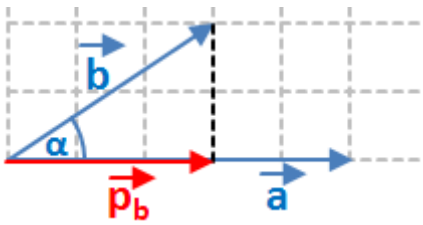


The dot product is $\mathbf{a} \cdot \mathbf{b} = a_x * b_x + a_y * b_y = 2 * 5 + 0 * 0 = 10$. This matches with $|a| * |b| * \cos\alpha = \sqrt{2^2} * \sqrt{5^2} * \cos 0° = 2 * 5 * 1 = 10$.

➔ If two vectors face the same direction, the dot product just the product of the length of the vectors.

3.2. Projection

Let's come to the interesting use case: The dot product can used to calculate the projection of one vector onto another vector:



The vector $\mathbf{p_b}$ is the projection of vector **b** onto vector **a**. We can derive the calculation of length of vector $\mathbf{p_b}$ as well as the vector itself using the dot product as follows:

Obviously, $\cos\alpha = \frac{|\mathbf{p_b}|}{|\mathbf{b}|}$, so $|\mathbf{p_b}| = \cos\alpha * |\mathbf{b}|$. Now let's do a trick and multiply both sides with the length of vector **a**:

$|\boldsymbol{a}| * |\mathbf{p_b}| = \cos\alpha * |\mathbf{a}| * |\mathbf{b}|$ => That's the definition of the dot product on the right side, thus

$$|\mathbf{p_b}| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|}.$$

So the length of vector $\mathbf{p_b}$ is the dot product of **a** and **b**, divided by the length of vector **a**.

How to get the vector $\mathbf{p_b}$ itself, not only its length? We see that **a** and $\mathbf{p_b}$ face into the same direction, so if both vectors are normalized (by dividing each of them by its length), then we get the same unit vector, that can be put on the same level, so it applies:

$$\frac{\boldsymbol{p_b}}{|\boldsymbol{p_b}|} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

$$\boldsymbol{p_b} = |\boldsymbol{p_b}| * \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Replace $|\mathbf{p_b}|$ with the equation above results in an equation to calculate $\mathbf{p_b}$ using the dot product:

$$\boldsymbol{p_b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|} * \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|^2} * \boldsymbol{a}$$

Example:

With vectors **a** = (5, 0) and **b** = (3, 2) from figure above:

Length of vector $\mathbf{p_b}$ is :

$$|\mathbf{p_b}| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|} = \frac{5 * 3 + 2 * 0}{\sqrt{5^2 + 0^2}} = \frac{15}{5} = 3.$$

Actual vector $\mathbf{p_b}$ is:

$$\boldsymbol{p_b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|^2} * \boldsymbol{a} = \frac{5 * 3 + 2 * 0}{5^2 + 0^2} * \binom{5}{0} = \frac{15}{25} * \binom{5}{0} = \binom{3}{0}$$

4. Summary

The definition of the dot product has been derived step by step, started from the calculation of the area of a rectangle.

Hope you liked it!

Visit my homepage: http://www.sunshine2k.de or http://www.bastian-molkenthin.de.

5. References

[1] **Right triangle @ Wikipedia**
[2] **Pythagorean theorem @ Wikipedia**
[3] **Dot Product @ Wikipedia**